ΠΑΡΑΔΕΙΓΜΑ 5:

$$2x^2 \cdot y'' + (3x+1)y' - y = 0$$

ΛΥΣΗ

Θέτω $\omega = \frac{1}{x}$

$$y' = \frac{dy}{dx} = \frac{dy}{dw}\cdot\frac{dw}{dx} = -\frac{1}{x^2}\cdot\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}\left(-w^2\frac{dy}{dw}\right) =$$

$$= -\frac{dy}{dx}\,w^2\cdot\frac{dy}{dw} - w^2\frac{dy}{dx}\left(\frac{dx}{dw}\right) =$$

$$= -\frac{dy}{dw}\,\frac{dw}{dx}(w^2)\frac{dy}{dw} + w^2\frac{dy}{dw}\left(\frac{dw}{dx}\cdot\frac{dy}{dw}\right)$$

- $y' = -w^2\frac{dy}{dw}$ (1)
- $y'' = w^4\frac{d^2y}{dw} + 2w^3\frac{dy}{dw}$ (2)

$$2w^2y'' + (w-w^2)\frac{dy}{dw} - y = 0 \quad , \quad w_0 = 0$$

Η ίδια εξίσωση του Παραδείγματος 1

άρα

$$\left.\begin{aligned} y_1(w) &= \sum_{n=0}^{\infty} c_n w^n = \sum_{n=0}^{\infty} c_n \frac{1}{x^n} \\ y_2(w) &= w^{-\frac{1}{2}}\cdot e^{w/2} = x^{1/2} e^{x/2} \end{aligned}\right\} , \; x \geq 0$$

Παράδειγμα 2:

$$x^2y'' - (x^2+x)y' + y = 0 \quad , \quad x_0 = 0$$

ΛΥΣΗ

$\alpha_2(x) = x^2$

$\alpha_2(x_0) = \alpha_2(0) = 0$

$\alpha_1(x) = -(x+x^2)$

$\alpha_0(x) = 1$

$$A_1(x) = x\,\frac{\alpha_1(x)}{\alpha_2(x)} = x\,\frac{(-x+x^2)}{x^2} = \frac{-x^2(1+x)}{x^2} =$$

$$= -1-x \rightsquigarrow p_0 = -1$$

$$A_2(x) = x^2\cdot\frac{\alpha_0(x)}{\alpha_2(x)} = x^2\frac{1}{x^2} = 1 \rightsquigarrow q_0 = 1$$

$$\lambda^2+(-1-1)\lambda+1=0$$
$$\lambda^2-2\lambda+1=0$$
$$\lambda_1=\lambda_2=1$$

$$y_1(x)=x\sum_{n=0}^{\infty}c_n\cdot x^n \rightsquigarrow y_1(x)=\sum_{n=0}^{\infty}c_n x^{n+1}, \quad c_0=1$$

$$0=x^2\cdot y''-x^2\cdot y_1'-xy_1'+y_1=$$

$$=x^2\cdot\sum_{n=0}^{\infty}c_n(n+1)n x^{n-1}-x^2\cdot\sum_{n=0}^{\infty}c_n(n+1)x^n-x\sum_{n=0}^{\infty}c_n(n+1)x^n+\sum_{n=0}^{\infty}c_n\cdot x^{n+1}=$$

$$=\sum_{n=0}^{\infty}c_n(n+1)n x^{n+1}-\sum_{n=0}^{\infty}c_n(n+1)x^{n+2}-\sum_{n=0}^{\infty}c_n(n+1)x^{n+1}+\sum_{n=0}^{\infty}c_n\cdot x^{n+1}=$$

$$=\sum_{n=0}^{\infty}c_n(n+1)n x^{n+1}-\sum_{n=1}^{\infty}c_{n-1}n x^{n+1}-\sum_{n=0}^{\infty}c_n(n+1)x^{n+1}+\sum_{n=0}^{\infty}c_n\cdot x^{n+1}=$$

$$=\sum_{n=1}^{\infty}c_n(n+1)n x^{n+1}-\sum_{n=1}^{\infty}c_{n-1}n x^{n+1}-\sum_{n=1}^{\infty}c_n(n+1)x^{n+1}+\sum_{n=1}^{\infty}c_n\cdot x^{n+1}=$$

$$=\sum_{n=1}^{\infty}\left(c_n(n+1)n-n\cdot c_{n-1}-c_n(n+1)+c_n\right)x^{n+1}=0 \Rightarrow$$

$$\Rightarrow c_n(n+1)n-n c_{n-1}-c_n(n+1)+c_n=0, \quad n\geq 1$$

$$\Rightarrow c_n=\frac{1}{n}c_{n-1}, \quad c_0=1, \quad n\geq 1 \rightsquigarrow c_n=\frac{1}{n!}, \quad n\geq 1$$

$$y_1(x)=x\left(x_0+\sum_{n=1}^{\infty}c_n x^n\right)=x\left(x_0+\sum_{n=1}^{\infty}\frac{x^n}{n!}\right)=x\cdot\sum_{n=0}^{\infty}\frac{x^n}{x!}=x\cdot e^x$$

Για τη δευτερη λυση είναι:

$$y_2(x)=y_1(x)\log x-x\sum_{n=0}^{\infty}d_n\cdot x^n, \quad d_0=0$$
$$y_2(x)=y_1(x)\log x-\sum_{n=0}^{\infty}d_n x^{n+1}$$
$$y_2'(x)=y_1'(x)\cdot\log x+\frac{1}{x}y_1(x)+\sum_{n=0}^{\infty}d_n(x+1)x^n$$
$$y_2''(x)=y_1''(x)\cdot\log x+2y_1'(x)\frac{1}{x}-y_1(x)\frac{1}{x^2}+\sum_{n=0}^{\infty}d_n(n+1)n\cdot x^{n-1}$$

Αντικαθιστώντας στην αρχική Δ.Ε, έχουμε:

$$2y_1'x-y_1+\sum_{n=0}^{\infty}d_n(n+1)n x^{n+1}-xy_1-\sum_{n=0}^{\infty}d_n(n+1)x^{n+2}-y_1-$$
$$-\sum_{n=0}^{\infty}d_n(n+1)x^{n+1}+\sum_{n=0}^{\infty}d_n x^{n+1}=0 \rightsquigarrow$$

$$\rightsquigarrow 2(e^x e^x)x-xe^x+\sum_{n=0}^{\infty}d_n(n+1)x^{n+1}-x^2e^x-\sum_{n=0}^{\infty}d_n(n+1)x^{n+2}$$
$$-x\cdot e^x-\sum_{n=0}^{\infty}d_n(n+1)x^{n+1}=0 \quad \text{όπου} \quad \sum_{n=1}^{\infty}\frac{x^n}{n!}=e^x.$$

$$\Rightarrow x \cdot \sum_{n=1}^{\infty} \left[n(n d_n - d_{n-1}) + \frac{1}{(n-1)!} \right] x^n = 0 \leadsto$$

$$\Rightarrow n d_n - d_{n-1} = -\frac{1}{n!} \Rightarrow d_n = \frac{1}{n} d_{n-1} - \frac{1}{n \cdot n!}, \ n \geq 1$$

Αρα, $y(x) = c_1 y_1 + c_2 y_2$.

Παραδειγμα 3: $x^2 \cdot y'' - x y' + 8(x^2-1) y = 0, \ x_0 = 0$

Λυση

$\alpha_2(x) = x^2, \ \alpha_1(x) = -x, \ \alpha_0(x) = -8 + 8x^2$

$\alpha_2(0) = 0 \leadsto$ Ανωμαλο σημειο

$A_1(x) = x \dfrac{\alpha_1(x)}{\alpha_2(x)} = x \dfrac{-x}{x^2} = -1 \longrightarrow p_0 = -1$

$A_0(x) = x^2 \dfrac{-8+8x^2}{x^2} = -8 + 8x^2 \longrightarrow q_0 = -8$

Αρα, $\lambda^2 + (-2\lambda) - 8 = 0 \rightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_2 = -2 \end{cases}$

$y_1(x) = \sum_{n=0}^{\infty} c_n \cdot x^{n+4}, \ c_0 = 1 \quad x \geq 0$

$c_0 = 1$

$c_1 = 0, \quad n(n+6) c_n = -8 c_{n-2}, \ n \geq 2$

$c_n = \dfrac{-8}{n(n+6)} c_{n-2}, \ n \geq 2$

Αφου, το βημα ειναι 2 τοτε ολοι οι περιττοι οροι αφου $c_1 = 0$ θα ειναι 0 δηλαδη: $c_{2n+1} = 0, \ n \geq 0$

Αυτο, το αναγραφιε οτι λαμβανοντας τις περιπτωσης $n = 2k$ και $n = 2k+1$

Αρα, ως τωρα εχουμε:

$c_0 = 1, \ c_{2n+1} = 0$ και $c_{2n} = ;$

$y_1(x) = 1 + \sum_{n=1}^{\infty} c_{2n} x^{2n}$

και

$$y_2(x) = c\, y_1(x) \log x + x^{-2} \cdot \sum_{n=0}^{\infty} d_n x^n, \ d_0 = 1$$

$$\frac{1}{x}\{ -5d_1 + 8(-d_2 + d_0)x + (-9d_3 + 8d_1)x^2 + 8(-d_4 + d_2)x^3 + (-5d_5 + 8d_3)x^4$$
$$+ \sum_{n=0}^{\infty} (n(n+6) d_{n+6} + 8d_{n-4} + 2c(n+3) c_n) x^{n+5} = 0$$

$$\left.\begin{array}{l} d_1 = 0 \\ -d_2 + d_0 = 0 \\ -9d_3 + 8d_1 = 0 \\ -d_4 + d_2 = 0 \\ -5d_5 + 8d_3 = 0 \end{array}\right\} \begin{array}{l} d_1 = d_3 = d_5 = 0 \\ d_0 = d_2 = d_4 = 1 \end{array}$$

και $n(n+6)\, d_{n+6} + 8d_{n-4} + 2C(n+3)C_n = 0$, $n \geq 0$.

για $n = 0$ παίρνουμε $8d_4 + 6C \cdot C_0 = 0 \rightsquigarrow C = -\frac{4}{3}$

Επειδή $C_{2n-1} = 0$ εύκολα βλέπουμε

$d_{2n-1} = 0$

Εδώ, είναι:

$$2n(n+3)\, d_{2n+6} + 4d_{2n+4} = -C(2n+3)\,\frac{2^{n+1} \cdot 3(-1)^n}{n! \cdot (n+3)!} \quad n = 1, 2, \ldots$$

Θέτοντας $d_6 = 0$

$$y_2(x) = x^4 \cdot \log|x| \cdot \sum_{n=0}^{\infty} \frac{2^{n+3} \cdot (-1)^{n+1}}{n!(n+3)!} x^{2n} + \frac{1}{2}\left(1 + x^2 + x^4 + \sum_{n=1}^{\infty} d_{2n+6} x^{2n+6}\right)_{x \neq 0}$$